UNITE DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

Rapports de Recherche

1992

ème anniversaire

N° 1807

Programme 5
Traitement du Signal,
Automatique et Productique

SUPERPOSITION PROPERTIES AND PERFORMANCE BOUNDS OF STOCHASTIC TIMED EVENT GRAPHS

Xiaolan XIE

Décembre 1992

RR-1807

Propriétés de Superposition et Bornes des Performances des Graphes d'Evénements Stochastiques

Xiaolan XIE
Projet SAGEP / INRIA-Lorraine
Technopôle Metz 2000, 4 rue Marconi, 57070 Metz, France

Résumé

Dans ce papier, nous étudions les performances des graphes d'événements stochastiques. Les temps de franchissement d'une transition sont générés par une séquence de variables aléatoires i.i.d. de distribution quelconque. Nous considérons d'abord un graphe d'événements stochastique dans lequel les temps de franchissement de chaque transition sont générés par la superposition (ou addition) de deux séquences de variables aléatoires. Nous mettons en évidence des propriétés importantes de ce système. Plus particulièrement, nous montrons que le temps de cycle moyen est sous-additif, c'est-à-dire qu'il est inférieur à la somme des temps de cycle des deux graphes d'événements obtenus en affectant à chaque transition une des deux séquences de variables aléatoires correspondants. A partir de ces propriétés, nous obtenons des bornes supérieures du temps de cycle moyen d'un graphe d'événements quelconque. En particulier, nous obtenons des bornes supérieures qui convergent vers le temps de cycle moyen exact lorsque les variances des temps de franchissement décroissent. Enfin, nous étudions les graphes d'événements dans lesquels les temps de franchissement sont bornés.

Mots-clefs : Réseaux de Petri, Graphes d'événements stochastiques, Bornes des performances

Superposition Properties and Performance Bounds of Stochastic Timed Event Graphs

Xiaolan XIE*

ABSTRACT

This paper addresses the performance evaluation of stochastic timed event graphs. The transition firing times are random variables with general distribution. We first consider a stochastic timed event graph in which the transition firing times are generated by the superposition (or addition) of two sets of random variables. Properties of this system are established. Especially, we prove that the average cycle time is sub-additive, i.e. it is smaller than the sum of the average cycle times of the two stochastic timed event graphs obtained by assigning to each transition one of the two related random variables. Based on these superposition properties, we derive various upper bounds of the average cycle time of a general stochastic timed event graph. Especially, we obtain upper bounds which converge to the exact average cycle time as the standard deviations decrease. Finally, we derive performance bounds for stochastic timed event graphs with bounded transition firing times.

KEYWORDS: Petri Nets, Stochastic Timed Event Graphs, Performance Bounds

* Mailing address : Dr XIE Xiaolan, SAGEP Project/INRIA, Technopôle Metz 2000, 4 rue Marconi, 57070 Metz, France. E-mail : xie@ilm.loria.fr Fax : (33) 87 76 39 77 Phone : (33) 87 20 35 00

1. INTRODUCTION

The dynamic behaviour of many real-life systems is characterized by synchronization, concurrency, common resources, etc. Tools are needed for modelling the dynamic behaviour of such systems, for checking the properties of the systems and for evaluating the system performances of interest such as throughput rate, queue lengths, waiting times and utilization ratios of resources. Petri nets provide a unified tool for system modelling, property checking and performance evaluation. Petri nets have been applied to communication systems, computer systems, manufacturing systems, etc. Excellent surveys can be found in [10, 15].

In this paper we limit ourselves to stochastic timed event graphs. An event graph, also called marked graph, is a Petri net in which each place has exactly one input transition and one output transition. A strongly connected event graph has some important properties, specifically: (i) the number of tokens in any elementary circuit is constant, and (ii) the system is deadlock free iff each elementary circuit contains at least one token (see for instance [4, 5, 7, 8]).

In the deterministic case, it has been proven [4, 11] that: (i) the cycle time of an elementary circuit is given by the ratio of the sum of the firing times of the transitions of the circuit by the number of tokens in the circuit; (ii) the cycle time of a strongly connected event graph is equal to the greatest cycle time among the ones of all the elementary circuits. Furthermore, a specified cycle time α being given, algorithms have been proposed in [9] to find an initial marking which leads to a cycle time less than α while minimizing a linear criterion which is a p-invariant.

In the stochastic case, it is no more possible to take advantage of the elementary circuits to evaluate the behaviour of the event graph and to reach a given performance. Previous work mainly focused on ergodicity conditions and performance bounds. Ergodicity conditions have been obtained for timed event graphs [1], for stochastic timed Petri nets [6] and for max-plus algebra models of stochastic discrete event systems [13].

For a strongly connected stochastic timed event graph, it has been proven that an average cycle time exists under some fairly weak conditions (see section 2). Both upper bounds and lower bounds have been proposed (see [2, 3, 11, 12]). The existing lower bounds are usually very tight and all of them are equal to the exact average cycle time in the deterministic case. Unfortunately, the upper bounds are usually loose and they are greater than the exact average cycle time even in the deterministic case.

The stochastic marking optimization problem, which consists in obtaining a specified cycle time while minimizing a linear criterion depending on the initial marking, has been addressed in [11, 12]. It has been proven that any cycle time greater than the maximal mean transition firing time can be reached provided that enough tokens are available. Heuristic algorithms for solving the stochastic marking optimization problem have been proposed.

The purpose of this paper is to introduce some superposition properties of stochastic timed event graphs which can be used to derive tight upper bounds. The firing times of any transition of a stochastic timed event graph are generated by a sequence of random variables with general distribution.

Section 3 considers a stochastic timed event graph in which the transition firing times are generated by the superposition (or addition) of two sets of random variable sequences. Properties of this system are established. Especially, we prove that the average cycle time is sub-additive, i.e. it is smaller than the sum of the average cycle times of the two stochastic timed event graphs in which the transition firing times are generated by one of the two sets of random variable sequences, respectively.

Based on these superposition properties, we derive in section 4 various upper bounds of the average cycle time of a general stochastic timed event graph. Especially, the mean values of the transition firing times being given, we provide an upper bound which depends on the standard deviations of the transition firing times. This upper bound converges to the exact average cycle time as the standard deviations tend to zero. Since the standard deviations are small in most real-life systems, this bound can be applied to provide a fast evaluation of the average cycle time.

Section 5 attempts to improve the upper bounds obtained in section 4. Tighter upper bounds are obtained. We further investigate three particular cases in which the transition firing times are random variables with uniform distribution in the first case, exponential distribution in the second case and normal distribution in the third case. Upper bounds are obtained for each case. These upper bounds show that the firing time randomness of transitions belonging to elementary circuits with small average cycle time has little impact on the average cycle time of the net.

Section 6 is devoted to stochastic timed event graphs in which the transition firing times are bounded. Characteristics and tighter upper bounds are provided. These upper bounds converge to the exact average cycle time as the bounds of transition firing times converge.

2. NOTATIONS AND ASSUMPTIONS

Let $N = (P, T, F)$ be the strongly connected event graph considered. P is the set of places, T is the set of transitions, and $F \subseteq (P \times T) \cup (T \times P)$ is the set of directed arcs. We denote by M_0 the initial marking of N.

Since N is an event graph, each place has exactly one input transition and one output transition. Without loss of generality, we assume that there exists at most one place between any two transitions. The following notations will be used :

$^\bullet t$ (resp. $t^\bullet$) : set of input (resp. output) places of transition t

$^\bullet p$ (resp. $p^\bullet$) : unique input (resp. output) transition of place p

in(t) : set of transitions which immediately precede transition t, i.e.

$$\mathrm{in}(t) = \{s \in T / \exists\, p \in P, {}^\bullet p = s \text{ and } p^\bullet = t\}$$

out(t) : set of transitions which immediately follow transition t, i.e.

$$\mathrm{out}(t) = \{s \in T / \exists\, p \in P, {}^\bullet p = t \text{ and } p^\bullet = s\}$$

(t, s) : place connecting transition t to transition s

Γ: set of elementary circuits of N

$M_0(\gamma)$: total number of tokens contained initially in $\gamma \in \Gamma$

We assume that no transition can be fired by more than one token at any time (i.e. recycled transitions), i.e. $(t, t) \in P$ and $M_0((t,t)) = 1$, $\forall\, t \in T$. We further assume that, when a transition fires, the related tokens remain in the input places until the firing process ends. They then disappear and one new token appears in each output place of the transition.

The following notations are used throughout this paper :

$X_t(k)$: nonnegative random variable related to the time required for the k-th firing of transition t

$S_t(k)$: instant of the k-th firing initiation of transition t

By convention, $X_t(k) = 0$, $\forall\, k \leq 0$ and $S_t(k) = 0$, $\forall\, k \leq 0$. As shown in [4], the transition firing initiation instants can be determined by the following recursive equations :

$$S_t(k) = \max_{\tau \in \mathrm{in}(t)} \left\{ S_\tau\big(k - M_0((\tau,t))\big) + X_\tau\big(k - M_0((\tau,t))\big) \right\} \tag{1}$$

We assume that the sequences of transition firing times $\{X_t(k)\}_{k=1}^{\infty}$ for $t \in T$ are mutually independent sequences of independent identically distributed (i.i.d.) integrable random variables.

It was proven in [1] that, under the foregoing assumptions, there exists a positive constant $\pi(M_0)$ such that:

$$\lim_{k\to\infty} S_t(k)/k = \lim_{k\to\infty} E\left[S_t(k)\right]/k = \pi(M_0), \quad \text{a.s.} \ \forall\, t \in T \tag{2}$$

$\pi\,(M_0)$ is the average cycle time of the event graph.

Since $\{X_t(k)\}_{k=1}^{\infty}$ are sequences of i.i.d. random variables, the index k is often omitted and we use X_t to denote the firing time of transition t whenever k is not necessary. We further assume that the first and second moments of X_t exist and denote by m_t its mean value and by σ_t its standard deviation, i.e. $m_t = E\,[X_t]$ and $\sigma_t^2 = E\,[(X_t - m_t)^2]$.

Since any stochastic timed event graph is completely characterized by its net structure, its initial marking and the set of transition firing time sequences, it can be denoted by the triplet $(N, M_0, \{X_t(k)\})$.

Finally, we assume that the stochastic timed event graph is deadlock free, which implies that each elementary circuit contains at least one token, i.e. $M_0(\gamma) \geq 1$, $\forall \gamma \in \Gamma$.

In the remainder of this section, we present some important properties of live strongly connected stochastic timed event graphs.

Property 1. (Commoner et *al.* [5])
In a strongly connected event graph, a marking M can be reached by firing from a live marking M' iff they have identical circuit counts, i.e. $M(\gamma) = M'(\gamma)$, $\forall \gamma \in \Gamma$.

Property 2. (Commoner et *al.* [5])
In a strongly connected event graph if a marking M' is reachable from a live marking M, then M is also reachable from M'.

Property 3.
In a strongly connected stochastic timed event graph if M is a marking reachable from a live marking M_0, then it holds that

$\pi(M) = \pi(M_0)$.

These properties claim that the two stochastic timed event graphs $(N, M_0, \{X_t(k)\})$ and $(N, M, \{X_t(k)\})$ have the same average cycle time if M and M_0 have the same circuit counts.

Proof of Property 3 :
Let us first prove that $\pi(M) \geq \pi(M_0)$. Since M is reachable from M_0, there exists a finite sequence of transitions $\upsilon \in T^*$ which leads to marking M from M_0, i.e. $M_0\,(\upsilon > M$.

Starting from M_0, consider the following operating mode. We first fire that first transition of the sequence υ. At the end of the firing of this transition, the second transition of υ is fired. At the end of the second firing, the third transition of υ is fired, and so on and so forth. The sequence υ is completely fired and the marking M is reached at instant ξ with

$$\xi = \sum_{t \in T} \sum_{k=1}^{\varpi_t} X_t(k)$$

where ϖ_t is the number of occurrences of transition t in the sequence υ. After instant ξ, the earliest operating mode which consists in firing the transitions as soon as they are firable is used.

Let $S_t^0(k)$ be the k-th firing initiation instant of transition t in this constrained operating mode. We also consider another stochastic timed event graph (N, M_0, $\{Y_t(k)\}$) with $Y_t(k) = X_t(k+\varpi_t)$ and let $S_t^1(k)$ be the k-th firing initiation instant of transition t in the earlist operating mode of this new system. From the above definition,

$$S_t^0(k + \varpi_t) = \xi + S_t^1(k), \qquad \forall k \geq 0, \forall t \in T$$

Since $\{X_t(k)\}_{k=1}^{\infty}$ are mutually independent sequences of i.i.d. random variables, we have :

$$\lim_{k\to\infty} \frac{S_t^1(k)}{k} = \lim_{k\to\infty} \frac{E\left[S_t^1(k)\right]}{k} = \pi(M), \qquad \forall t \in T$$

Combining the above two relations,

$$\lim_{k\to\infty} \frac{S_t^0(k)}{k} = \lim_{k\to\infty} \frac{S_t^0(k)}{k + \varpi_t} = \lim_{k\to\infty} \left[\frac{\xi}{k + \varpi_t} + \frac{k}{k + \varpi_t} \frac{S_t^1(k)}{k_t}\right] = \pi(M), \qquad \forall t \in T$$

Starting again from M_0 and using the earliest operating mode, the average cycle time is equal to $\pi(M_0)$ and the k-th firing initiation instant of t is $S_t(k)$. As shown in [4],

$$S_t(k) \leq S_t^0(k), \qquad \forall k \geq 0, \forall t \in T$$

which leads to :

$$\lim_{k\to\infty} \frac{S_t(k)}{k} \leq \lim_{k\to\infty} \frac{S_t^0(k)}{k}, \qquad \forall t \in T$$

or :

$$\pi(M_0) \leq \pi(M)$$

Thanks to Property 2, M_0 is also reachable from M. Using similar arguments than above, it can be shown that

$$\pi(M) \leq \pi(M_0)$$

which finally implies that

$\pi(M) = \pi(M_0)$

Q.E.D.

3. SUPERPOSITION PROPERTIES

This section presents some superposition properties of stochastic timed event graphs. We first consider a stochastic timed event graph in which the transition firing times are generated by the superposition of two sets of random variable sequences. The firing initiation instants are shown to be sub-additive. From this property, the superposition properties of some long term performance measures are established. Finally, these properties are generalized to the case where the transition firing times are generated by the linear combination of several sets of random variable sequences.

3.1. Superposition property of the firing initiation instants

Let us consider two sets of non-negative random variable sequences $\left\{X_t^1(k)\right\}_{k=0}^{\infty}$ and $\left\{X_t^2(k)\right\}_{k=0}^{\infty}$. Consider the stochastic timed event graph STEG = (N, M_0, $\{X_t(k)\}$) with

$$X_t(k) = X_t^1(k) + X_t^2(k), \quad \forall t \in T, \forall k \tag{3}$$

Let us also consider two stochastic timed event graphs STEG1 and STEG 2 defined as follows:

STEG1 = (N, M_0, $\{X_t^1(k)\}$) and STEG2 = (N, M_0, $\{X_t^2(k)\}$)

These two stochastic timed event graphs have the same net structure and the same initial marking as STEG. However, the transition firing times of STEG1 are generated by $\{X_t^1(k)\}$ while those of STEG2 are generated by $\{X_t^2(k)\}$.

Let $S_t^1(k)$ (resp. $S_t^2(k)$) be the instant of the k-th firing initiation of transition t in STEG1 (resp. STEG2). The following property claims that the firing initiation instants are sub-additive.

Theorem 1.

$$S_t(k) \le S_t^1(k) + S_t^2(k), \quad \forall t \text{ and } \forall k \tag{4}$$

Proof of Theorem 1 :

The proof is based on equation (4) and it is done by induction on (t, k). First, equation is clearly true for all t and for all $k \le 0$. Let us assume that relation (4) holds up to (t, k) at the exclusion of this point.

By equation (1),

$$S_t(k) = \underset{\tau \in in(t)}{Max}\left\{S_\tau\left(k - M_0((\tau,t))\right) + X_\tau\left(k - M_0((\tau,t))\right)\right\} \tag{5}$$

By induction assumption,

$$S_\tau(k') \le S_\tau^1(k') + S_\tau^2(k'), \quad \forall\tau \in in(t) \text{ and } \forall k' \le k$$

which leads to

$$S_\tau\left(k - M_0((\tau,t))\right) \le S_\tau^1\left(k - M_0((\tau,t))\right) + S_\tau^2\left(k - M_0((\tau,t))\right), \quad \forall\tau \in in(t)$$

Combining with relations (3) and (5),

$$S_t(k) \le \underset{\tau \in in(t)}{Max}\left\{S_\tau^1\left(k^*\right) + S_\tau^2\left(k^*\right) + X_\tau^1\left(k^*\right) + X_\tau^2\left(k^*\right)\right\}$$

where $k^* = k - M_0((\tau, t))$. This leads to :

$$S_t(k) \le \underset{\tau \in in(t)}{Max}\left\{S_\tau^1\left(k^*\right) + X_\tau^1\left(k^*\right)\right\} + \underset{\tau \in in(t)}{Max}\left\{S_\tau^2\left(k^*\right) + X_\tau^2\left(k^*\right)\right\}$$

By equation (1), we obtain

$$S_t(k) \le S_t^1(k) + S_t^2(k)$$

Q.E.D.

As it can be noticed, Theorem 1 is very general and no assumption about the transition firing times is needed. Especially, the transition firing times are not needed to be mutually independent, stationary and ergodic.

3.2. Superposition properties of long term performance measures

In this subsection, we consider the three stochastic timed event graphs defined in subsection 3.1. The purpose of this subsection is to establish the superposition properties of three long term performance measures : the average cycle time, the queue lengths and the utilization ratios.

We first consider the average cycle time. Let us assume that the average cycle time of both STEG1 and STEG2 exists. Let $\pi^1(M_0)$ (resp. $\pi^2(M_0)$) be the average cycle time of STEG1 (resp. STEG2). The following theorem claims that the average cycle time of STEG is also sub-additive.

Theorem 2.

$$\pi(M_0) \le \pi^1(M_0) + \pi^2(M_0) \tag{6}$$

Proof of Theorem 2 :

From Theorem 1,

$$\frac{S_t(k)}{k} \le \frac{S_t^1(k)}{k} + \frac{S_t^2(k)}{k}$$

By letting $k \to \infty$ and from the ergodicity of the stochastic timed event graphs,

$$\pi(M_0) \le \pi^1(M_0) + \pi^2(M_0)$$

Q.E.D.

Consider now the other performance measures of STEG. For each place p, let $\omega(M_0,p)$ be the average number of tokens waiting in place p (not including the one being served). For each transition t, let $\upsilon(M_0,t)$ be the utilization ratio of transition t. These two measures can be determined as follows :

$$\omega(M_0,p) = \lim_{H\to\infty} \frac{1}{H}\int_{s=0}^{H}\left(M_s(p) - \alpha_{p\bullet}(s)\right)ds \tag{7}$$

and

$$\upsilon(M_0,t) = \lim_{H\to\infty} \frac{1}{H}\int_{s=0}^{H}\alpha_t(s)\,ds \tag{8}$$

where M_s is the marking at instant s, $\alpha_t(s) = 1$ if transition t is busy at instant s and $\alpha_t(s) = 0$ otherwise.

Let us first derive some relations between the three performance measures which will be used for deriving other superposition properties. From the definition of the utilization ratios, it holds that

$$\upsilon(M_0,t) = \frac{m_t}{\pi(M_0)} \tag{9}$$

From the definitions of average cycle time $\pi(M_0)$ and $\omega(M_0,p)$, in any time period of length $\pi(M_0)$ of the steady state, $\omega(M_0,p) * \pi(M_0)$ is the average accumulated waiting time of tokens in place p and m_t is the average busy time of transition t. Since the total number of tokens in any elementary circuit remains invariant whatever the transition firings, it holds that

$$\sum_{p\in\gamma}\omega(M_0,p)\pi(M_0) + \sum_{t\in\gamma} m_t = M_0(\gamma)\pi(M_0), \quad \forall\gamma\in\Gamma$$

which leads to :

$$\omega(M_0,\gamma)\pi(M_0) + \sum\nolimits_{t\in\gamma} m_t = M_0(\gamma)\pi(M_0), \quad \forall\gamma\in\Gamma \tag{10}$$

where

$$\omega(M_0,\gamma) = \sum\nolimits_{p\in\gamma}\omega(M_0,p) \tag{11}$$

We can now introduce the superposition properties of average queue lengths and utilization ratios. For each place p, let $\omega^1(M_0,p)$ (resp. $\omega^2(M_0,p)$) be the average number of tokens waiting in place p of STEG1 (resp. STEG2). For each transition t, let $\upsilon^1(M_0,t)$ (resp. $\upsilon^2(M_0,t)$) be the utilization ratio of transition t of STEG1 (resp. STEG2).

Theorem 3.

(a) $$\frac{m_t}{\upsilon(M_0,t)} \le \frac{m_t^1}{\upsilon^1(M_0,t)} + \frac{m_t^2}{\upsilon^2(M_0,t)}, \quad \forall t \tag{12}$$

(b) $$\omega(M_0,\gamma)\,\pi(M_0) \le \omega^1(M_0,\gamma)\,\pi^1(M_0) + \omega^2(M_0,\gamma)\,\pi^2(M_0) \qquad \forall \gamma \in \Gamma \tag{13}$$

Proof of Theorem 3 :

From relation (9),

$$\pi(M_0) = \frac{m_t}{\upsilon(M_0,t)},\ \pi^1(M_0) = \frac{m_t^1}{\upsilon^1(M_0,t)} \text{ and } \pi^2(M_0) = \frac{m_t^2}{\upsilon^2(M_0,t)}$$

By replacing these terms in Theorem 2, property (a) can be proved.

Now let us prove property (b). From relation (3),

$$m_t = m_t^1 + m_t^2, \qquad \forall t$$

Replacing it in relation (10), we obtain

$$\omega(M_0,\gamma)\pi(M_0) = M_0(\gamma)\pi(M_0) - \sum_{t\in\gamma} m_t^1 - \sum_{t\in\gamma} m_t^2$$

From theorem 2,

$$\omega(M_0,\gamma)\pi(M_0) \le M_0(\gamma)\pi^1(M_0) - \sum_{t\in\gamma} m_t^1 + M_0(\gamma)\pi^2(M_0) - \sum_{t\in\gamma} m_t^2$$

Applying relation (10) to STEG1 and STEG2, we obtain :

$$\omega(M_0,\gamma)\,\pi(M_0) \le \omega^1(M_0,\gamma)\,\pi^1(M_0) + \omega^2(M_0,\gamma)\,\pi^2(M_0)$$

Q.E.D.

From theorem 3, the following corollary can be easily proven.

Corollary 1.

(a) $\upsilon(M_0,t) \ge \mathrm{Min}\{\upsilon^1(M_0,t), \upsilon^2(M_0,t)\}, \qquad \forall t \in T$

(b) $\omega(M_0,\gamma) \le \omega^1(M_0,\gamma) + \omega^2(M_0,\gamma), \qquad \forall \gamma \in \Gamma$

3.3. Generalization

Instead of the superposition of two sets of random variable sequences, this section considers a stochastic timed event graph in which the transition firing times are generated by the linear combination of several sets of random variable sequences.

Let us consider n sets of non-negative random variable sequences $\left\{X_t^i(k)\right\}_{k=0}^{\infty}$ and n positive coefficients α_i for i = 1, 2, ..., n. Consider the stochastic timed event graph STEG = (N, M_0, $\{X_t(k)\}$) with

$$X_t(k) = \sum_{i=1}^{n} \alpha_i X_t^i(k), \quad \forall t \in T, \forall k \tag{14}$$

We give without proof the superposition properties of this stochastic timed event graph :

Theorem 4.

$$\text{(a) } S_t(k) \leq \sum_{i=1}^{n} \alpha_i S_t^i(k), \quad \forall t \in T, \forall k \tag{15}$$

$$\text{(b) } \pi(M_0) \leq \sum_{i=1}^{n} \alpha_i \pi^i(M_0) \tag{16}$$

$$\text{(c) } \frac{m_t}{\upsilon(M_0,t)} \leq \sum_{i=1}^{n} \alpha_i \frac{m_t^i}{\upsilon^i(M_0,t)}, \qquad \forall t \tag{17}$$

$$\text{(d) } \pi(M_0)\omega(M_0,\gamma) \leq \sum_{i=1}^{n} \alpha_i \pi^i(M_0)\omega^i(M_0,\gamma), \qquad \forall \gamma \in \Gamma \tag{18}$$

Corollary 2.

(a) $\upsilon(M_0,t) \geq \text{Min}\{\upsilon^i(M_0,t) \text{ for } i = 1, 2, \ldots, n\}, \qquad \forall t \in T$

(b) $\omega(M_0,\gamma) \leq \omega^1(M_0,\gamma) + \ldots + \omega^n(M_0,\gamma), \qquad \forall \gamma \in \Gamma$

4. UPPER BOUNDS OF THE AVERAGE CYCLE TIME

In this section, we assume that both the first and the second moments of the transition firing times exist. The purpose here is to derive upper bounds of the average cycle time $\pi(M_0)$ by using the superposition properties developed in the previous section.

To this end, let us consider two sets of random variable sequences $\left\{X_t^1(k)\right\}_{k=0}^{\infty}$ and $\left\{X_t^2(k)\right\}_{k=0}^{\infty}$ defined as follows :

$$X_t^1(k) = E[X_t(k)] = m_t$$
$$X_t^2(k) = (X_t(k) - m_t)^+$$

Consider two stochastic timed event graphs STEG1 = (N, M_0, $\{X_t^1(k)\}$) and STEG2 = (N, M_0, $\{X_t^2(k)\}$). The assumptions of section 2 guarantee that the average cycle times of these two stochastic timed event graphs exist. Let $\pi^1(M_0)$ be the average cycle time of STEG1 and $\pi^2(M_0)$ the average cycle time of STEG2.

As shown in [4, 13],

$$\pi^1(M_0) = \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} \tag{19}$$

Let us now consider a new stochastic timed event graph STEG3 = (N, M_0, $\{X_t^3(k)\}$) with

$$X_t^3(k) = X_t^1(k) + X_t^2(k), \quad \forall t \in T, \forall k$$

The assumptions of section 2 ensure the existence of the average cycle time of STEG3 and we denote it by $\pi^3(M_0)$. From Theorem 2,

$$\pi^3(M_0) \le \pi^1(M_0) + \pi^2(M_0) \tag{20}$$

Since for all t and for all k,

$$X_t^3(k) = X_t^1(k) + X_t^2(k) = m_t + \left(X_t(k) - m_t\right)^+ \ge X_t(k)$$

it holds that

$$\pi(M_0) \le \pi^3(M_0)$$

Combining with relations (19) and (20), we obtain

$$\pi(M_0) \le \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} + \pi^2(M_0) \tag{21}$$

The RHS term of relation (21) is an upper bound of the average cycle time. Its first term depends only on the first moment of the transition firing times while the second term depends on the higher moments. The following theorem shows that the second term is in turn upper bounded by the sum of the standard deviations.

Theorem 5.

$$\pi(M_0) \le \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} + \sum_{t\in T} \sigma_t \tag{22}$$

Proof of Theorem 5 :

Thanks to relation (21), we only need to prove that

$$\pi^2(M_0) \le \sum_{t\in T} \sigma_t$$

The proof is based on the upper bound proposed by Campos et *al.* [3]. Since the event graph is strongly connected and the transitions are recycled,

$$\pi^2(M_0) \le \sum_{t \in T} E\left[X_t^2\right] \tag{23}$$

Since

$$E\left[\left(X_t^2 - E\left[X_t^2\right]\right)^2\right] = E\left[\left(X_t^2\right)^2\right] - \left(E\left[X_t^2\right]\right)^2 \ge 0$$

it holds that :

$$\left(E\left[X_t^2\right]\right)^2 \le E\left[\left(X_t^2\right)^2\right] = E\left[\left((X_t - m_t)^+\right)^2\right] \le E\left[(X_t - m_t)^2\right] = (\sigma_t)^2$$

which yields that :

$$E\left[X_t^2\right] \le \sigma_t$$

Combining with relation (23),

$$\pi^2(M_0) \le \sum_{t \in T} \sigma_t$$

Q.E.D.

As it can be noticed, this upper bound decreases as the standard deviations decrease. The following theorem shows that it converges to the exact average cycle time which is equal to the average cycle time of the related deterministic timed event graph. Since the standard deviations are small in most real-life systems, this upper bound can be used to provide a fast performance evaluation of a system subject to perturbations.

Theorem 6.

Let $\sigma = \sum_{t \in T} \sigma_t$. It holds that :

$$\lim_{\sigma \to 0} \pi(M_0) = \operatorname*{Max}_{\gamma \in \Gamma} \frac{\sum_{t \in \gamma} m_t}{M_0(\gamma)}$$

Proof of Theorem 6 :

As shown in [2, 3],

$$\pi(M_0) \ge \operatorname*{Max}_{\gamma \in \Gamma} \frac{\sum_{t \in \gamma} m_t}{M_0(\gamma)}$$

Combining with Theorem 5, we obtain :

$$\lim_{\sigma \to 0} \pi(M_0) = \operatorname*{Max}_{\gamma \in \Gamma} \frac{\sum_{t \in \gamma} m_t}{M_0(\gamma)}$$

Q.E.D.

The following corollary is a generalization of Theorem 6. It shows that the average cycle time of a stochastic timed event graph converges to that of another stochastic timed event graph if the transition firing times of the first stochastic timed event graph converge to those of the second stochastic timed event graph in both the first and second moments.

Corollary 3.
Let $\pi^0(M_0)$ be the average cycle time of the stochastic timed event graph STEG0 = (N, M_0, $\{X_t^0(k)\}$) and let $s = \sum_{t\in T} \mathrm{Var}\left(X_t - X_t^0\right)$. Then it holds that :

$$\lim_{s\to 0} \pi(M_0) = \pi^0(M_0)$$

Proof of Corollary 3 :
We first show that :

$$\lim_{s\to 0} \pi(M_0) \le \pi^0(M_0)$$

To this end, consider two sets of random variable sequences $\left\{X_t^1(k)\right\}_{k=0}^{\infty}$ and $\left\{X_t^2(k)\right\}_{k=0}^{\infty}$ defined as follows :

$$X_t^1(k) = X_t^0(k)$$
$$X_t^2(k) = \left(X_t(k) - X_t^0(k)\right)^+$$

As in the proof of Theorem 5, it can be shown that :

$$\pi(M_0) \le \pi^0(M_0) + \sum_{t\in T}\sqrt{\mathrm{Var}\left(X_t - X_t^0\right)}$$

Which implies that :

$$\lim_{s\to 0} \pi(M_0) \le \pi^0(M_0)$$

We then show that :

$$\lim_{s\to 0} \pi(M_0) \ge \pi^0(M_0)$$

To this end, consider two other sets of random variable sequences $\left\{X_t^1(k)\right\}_{k=0}^{\infty}$ and $\left\{X_t^2(k)\right\}_{k=0}^{\infty}$ defined as follows :

$$X_t^1(k) = X_t(k)$$
$$X_t^2(k) = \left(X_t^0(k) - X_t(k)\right)^+$$

As in the proof of Theorem 5, it can be shown that :

$$\pi^0(M_0) \le \pi(M_0) + \sum_{t\in T}\sqrt{\mathrm{Var}\left(X_t - X_t^0\right)}$$

Which implies that :

$$\lim_{s\to 0} \pi(M_0) \ge \pi^0(M_0)$$

Q.E.D.

5. IMPROVING THE UPPER BOUNDS

This section attempts to improve the previous bound by adequately choosing the two random variable sequences which have been used to derive the upper bound of theorem 5.

Let us consider again two sets of random variable sequences $\left\{X_t^1(k)\right\}_{k=0}^{\infty}$ and $\left\{X_t^2(k)\right\}_{k=0}^{\infty}$ defined as follows :

$$X_t^1(k) = z_t$$
$$X_t^2(k) = \left(X_t(k) - z_t\right)^+$$

where z_t is a non-negative constant.

Let $\pi^1(M_0, Z)$ be the average cycle time of STEG1 = $(N, M_0, \{X_t^1(k)\})$ and $\pi^2(M_0, Z)$ the average cycle time of STEG2 = $(N, M_0, \{X_t^2(k)\})$ where Z denotes the vector of constants zt, i.e. $Z = (z_1\ z_2 \dots z_t \dots z_K)^T$ where K = card(T).

As in section 4, it can be shown that

$$\pi(M_0,Z) \le \pi^1(M_0,Z) + \pi^2(M_0,Z) \tag{24}$$

where the two RHS terms satisfy the following relations :

$$\pi^1(M_0,Z) = \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} z_t}{M_0(\gamma)} \tag{25}$$

$$\pi^2(M_0,Z) \le \sum_{t\in T} E\left[\left(X_t - z_t\right)^+\right] \tag{26}$$

Since relation (24) holds for all $z_t \ge 0$, the smallest upper bound can be obtained by adequately choosing the value of z_t as shown in the following theorem.

Theorem 7.

$$\text{(a) } \pi(M_0) \le \operatorname*{Min}_{z_t\ge 0}\left\{ \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} z_t}{M_0(\gamma)} + \pi^2(M_0,Z)\right\} \tag{27}$$

$$\text{(b) } \pi(M_0) \le \underset{z_t \ge 0}{\mathrm{Min}} \left\{ \underset{\gamma \in \Gamma}{\mathrm{Max}} \frac{\sum_{t \in \gamma} z_t}{M_0(\gamma)} + \sum_{t \in T} E\left[(X_t - z_t)^+\right] \right\} \tag{28}$$

This upper bound can be easily obtained by solving a linear programming problem. Since the upper bounds of section 4 can be obtained by taking $z_t = m_t$, the new upper bounds are better than the upper bounds of section 4. As a result, it also converges to the exact average cycle time as the standard deviations decreases.

Moreover, these new bounds also shows that the firing time randomness of transitions belonging to elementary circuits with small average cycle times has little effect on the average cycle time of the whole system. The effect of their randomness can almost be completely canceled by taking large values for z_t.

The remainder of this section is devoted to three special cases where the transition firing times are random variables with uniform distribution in the first case, exponential distribution in the second case and normal distribution in the third case. Upper bounds are given in the following corollaries.

Corollary 4.
If the transition firing times X_t for all $t \in T$ are random variables with uniform distribution defined on $[a_t, b_t]$, then it holds that :

$$\text{(a) } \pi(M_0) \le \underset{\gamma \in \Gamma}{\mathrm{Max}} \frac{\sum_{t \in \gamma} (b_t + a_t)}{2M_0(\gamma)} + \sum_{t \in T} \frac{b_t - a_t}{8} \tag{29}$$

$$\text{(b) } \pi(M_0) \le \underset{a_t \le z_t \le b_t}{\mathrm{Min}} \left\{ \underset{\gamma \in \Gamma}{\mathrm{Max}} \frac{\sum_{t \in \gamma} z_t}{M_0(\gamma)} + \sum_{t \in T} \frac{(b_t - z_t)^2}{2(b_t - a_t)} \right\} \tag{30}$$

Proof of Corollary 4 :
As claim (a) can be obtained from claim (b) by taking $z_t = m_t$ (i.e. $z_t = (a_t + b_t)/2$), we only need to prove (b).

Since X_t are random variables with uniform distribution, it holds that for all $z_t \in [a_t, b_t]$

$$\begin{aligned} E\left[(X_t - z_t)^+\right] &= E\left[(X_t - z_t) \ / \ X_t \ge z_t\right] * P\{X_t \ge z_t\} \\ &= \frac{b_t - z_t}{2} * \frac{b_t - z_t}{b_t - a_t} \\ &= \frac{(b_t - z_t)^2}{2(b_t - a_t)} \end{aligned}$$

Combining with relation (b) of Theorem (6), the claim (b) is proved.

Q.E.D.

Corollary 5.

If the transition firing times X_t for all $t \in T$ are random variables with exponential distribution, then it holds that :

$$(a)\ \pi(M_0) \le \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} + \exp(-1) \sum_{t\in T} m_t \tag{31}$$

$$(b)\ \pi(M_0) \le \underset{z_t\ge 0}{\mathrm{Min}} \left\{ \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} z_t}{M_0(\gamma)} + \sum_{t\in T} m_t \exp\left(-\frac{z_t}{m_t}\right) \right\} \tag{32}$$

Proof of Corollary 5 :

The proof is similar to that of Corollary 4 and we only need to prove that :

$$E\left[(X_t - z_t)^+\right] = m_t \exp\left(-\frac{z_t}{m_t}\right)$$

Since X_t has exponential distribution,

$$E\left[(X_t - z_t)^+\right] = E\left[(X_t - z_t)\ /\ X_t \ge z_t\right] * P\{X_t \ge z_t\} = m_t * \exp\left(-\frac{z_t}{m_t}\right)$$

Q.E.D.

As the random variables with normal distribution may not be positive, the transition firing times in the third case are generated as follows :

$$X_t = \left(X_t^*\right)^+ \tag{33}$$

where X_t^* is a random variable with normal distribution with parameters (m_t, σ_t). We assume that $m_t/\sigma_t >> 1$ which ensures that X_t and X_t^* have almost the same probability distribution.

In the following, we denote by $\Phi(x)$ the standard normal distribution function and by $\phi(x)$ its density, i.e.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \qquad \Phi(x) = \int_{-\infty}^{x} \phi(x) ds \tag{34}$$

Corollary 6.

If the transition firing times X_t for all $t \in T$ are random variables defined by relation (33), then it holds that :

$$\text{(a) } \pi(M_0) \le \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} + \frac{1}{\sqrt{2\pi}} \sum_{t\in T} \sigma_t \tag{35}$$

$$\text{(b) } \pi(M_0) \le \operatorname*{Min}_{z_t\ge 0} \left\{ \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} z_t}{M_0(\gamma)} + \sum_{t\in T} \sigma_t f\left(\frac{z_t - m_t}{\sigma_t}\right) \right\} \tag{36}$$

where $f(x) = \phi(x) - x\,(1 - \Phi(x))$

Before giving the proof, let $z_t = m_t + n\,\sigma_t$ and consider the related bounds. In this case, relation (b) of Corollary 5 leads to :

$$\pi(M_0) \le \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} (m_t + n\sigma_t)}{M_0(\gamma)} + \sum_{t\in T} \sigma_t f(n) \tag{37}$$

Let us observe the values of function f(.) given in table 1. We notice that the value of f(.) converges rapidly to zero.

Table 1 : The values of function f(•)

n	0	1	2	3	4
f(•)	0.398	0.083	0.008	0.001	0.000

Taking n = 3, we obtain a very tight bound :

$$\pi(M_0) \le \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} (m_t + 3\sigma_t)}{M_0(\gamma)} + 0.001 \sum_{t\in T} \sigma_t \tag{38}$$

As the second term of this upper bound is small enough in real-life systems, this new upper bound further confirms that the firing time randomness of transitions belonging to elementary circuit with small average cycle time has little impact on the cycle time of the whole system.

Moreover, as σ_t are usually small compared to m_t, this upper bound proves the conjecture which claims that the average cycle time of a stochastic timed event graph mainly depends on the mean transition firing times.

Proof of Corollary 6 :

The proof is similar to that of Corollary 4 and we only need to prove that :

$$E\left[(X_t - z_t)^+\right] = \sigma_t f\left(\frac{z_t - m_t}{\sigma_t}\right)$$

Since the means of X_t is m_t and its standard deviation is σ_t, its density function is given by

$$\frac{1}{\sigma_t}\phi\left(\frac{x-m_t}{\sigma_t}\right)$$

Therefore,

$$E\left[(X_t-z_t)^+\right] = \int_{z_t}^{+\infty}(x-z_t)\frac{1}{\sigma_t}\phi\left(\frac{x-m_t}{\sigma_t}\right)dx$$

$$= \int_0^{+\infty}x\frac{1}{\sigma_t}\phi\left(\frac{x-m_t+z_t}{\sigma_t}\right)dx$$

$$= \sigma_t\int_0^{+\infty}\phi\left(\frac{x-m_t+z_t}{\sigma_t}\right)d\left(\frac{1}{2}\left(\frac{x-m_t+z_t}{\sigma_t}\right)^2\right)$$

$$+\int_0^{+\infty}\frac{m_t-z_t}{\sigma_t}\phi\left(\frac{x-m_t+z_t}{\sigma_t}\right)dx$$

Taking into account the definition of ϕ, we obtain :

$$E\left[(X_t-z_t)^+\right] = \sigma_t\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-m_t+z_t}{\sigma_t}\right)^2\right)d\left(\frac{1}{2}\left(\frac{x-m_t+z_t}{\sigma_t}\right)^2\right)$$

$$+(m_t-z_t)\int_0^{+\infty}\phi\left(\frac{x-m_t+z_t}{\sigma_t}\right)d\left(\frac{x-m_t+z_t}{\sigma_t}\right)$$

$$= \sigma_t\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{z_t-m_t}{\sigma_t}\right)^2\right)+(m_t-z_t)\left(1-\Phi\left(\frac{z_t-m_t}{\sigma_t}\right)\right)$$

$$= \sigma_t f\left(\frac{z_t-m_t}{\sigma_t}\right)$$

Q.E.D.

6. CASE OF BOUNDED TRANSITION FIRING TIMES

In this section, we consider stochastic timed event graphs with bounded transition firing times. Characteristics and performance bounds are presented.

Let $a_t \geq 0$ be the lower bound of the firing times of transition t and $b_t \geq 0$ the upper bound, i.e.

$$a_t \leq X_t(k) \leq b_t, \ \forall t, \forall k$$

For this stochastic timed event graph, some results are immediately available as shown in the following property.

Property 4.

$$(a)\ \pi(M_0) \geq \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} \geq \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} a_t}{M_0(\gamma)} \tag{39}$$

$$(b)\ \pi(M_0) \leq \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} b_t}{M_0(\gamma)} \tag{40}$$

$$(c)\ \pi(M_0) \leq \underset{\gamma\in\Gamma}{\mathrm{Max}} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} + \sum_{t\in T} \sigma_t \tag{41}$$

The purpose of the remainder of this section is to derive new upper bounds based on the stochastic comparison properties proposed by Baccelli and Liu [2] and the superposition properties presented above.

For this purpose, we consider random variables $Y_t(k) \in \{a_t, b_t\}$ defined as follows :

$$\Pr\{Y_t(k) = a_t\} = p_t \text{ and } \Pr\{Y_t(k) = b_t\} = 1 - p_t$$

where $p_t = \dfrac{b_t - m_t}{b_t - a_t}$

Lemma 1.

(a) $E[Y_t(k)] = m_t, \quad \forall t, \forall k$

(b) $X_t(k) \leq_{icx} Y_t(k), \quad \forall t, \forall k$

(c) $\mathrm{Var}(X_t(k)) = (\sigma_t)^2 \leq \mathrm{Var}(Y_t(k)) = (b_t - m_t)(m_t - a_t) \leq \left(\dfrac{b_t - a_t}{2}\right)^2, \ \forall t$

In this lemma, $\leq_{icx}$ denotes the convex ordering relation and Var(•) is the variance of the related random variable. This lemma shows that the random variables $X_t(k)$ and $Y_t(k)$ have the same mean and that $Y_t(k)$ is stochastically more variable than $X_t(k)$.

Proof of Lemma 1 :

Since claim (a) is obviously true and claim (c) follows directly from (b), we need only to prove claim (b).

Let F be the probability distribution function of $X_t(k)$ and G be the one of $Y_t(k)$. These two probability distribution functions are defined as follows :

$$G(y) = \begin{cases} 0, & \text{if } y < a_t \\ p_t, & \text{if } a_t \leq y < b_t \\ 1, & \text{if } y \geq b_t \end{cases} \quad \text{and} \quad F(x) = \begin{cases} 0, & \text{if } x < a_t \\ F(x), & \text{if } a_t \leq x < b_t \\ 1, & \text{if } x \geq b_t \end{cases}$$

Since F and G are identical everywhere except in the interval $[a_t, b_t[$, since G remains constant in this interval and since F is a non-decreasing function, there exists $\xi \in [a_t, b_t]$ such that

$$F(x) \le G(x) \ \ \forall x < \xi$$
$$F(x) \ge G(x) \ \ \forall x > \xi$$

which means that the random variables $X_t(k)$ and $Y_t(k)$ satisfy the cut criterion of Karlin and Novikoff (see [16]). As a result, $X_t(k) \le_{icx} Y_t(k)$.

Q.E.D.

Let $\pi^*(M_0)$ be the average cycle time of the stochastic timed event graph STEG*=(N, M_0, $\{Y_t(k)\}$). The following theorem shows that $\pi^*(M_0)$ is an upper bound of $\pi(M_0)$.

Theorem 8.

$$\pi(M_0) \le \pi^*(M_0) \tag{42}$$

Proof of Theorem 8 :

Since $\{X_t(k)\}_{k=1}^{\infty}$ and $\{Y_t(k)\}_{k=1}^{\infty}$ for all $t \in T$ are mutually independent sequences of i.i.d. random variables, Lemma 1 and Corollary 5.1. of [2] imply :

$$\pi^*(M_0) \ge \pi(M_0)$$

Q.E.D.

In the following we derive upper bounds of $\pi^*(M_0)$ which are also upper bounds of $\pi(M_0)$. The following theorem presents the first upper bound which is similar to the upper bound of Theorem 5.

Theorem 9.

$$\pi(M_0) \le \pi^*(M_0) \le \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} + \sum_{t\in T} \frac{(b_t - m_t)(m_t - a_t)}{b_t - a_t} \tag{43}$$

Proof of Theorem 9 :

As in the proof of Theorem 5, it can be shown that

$$\pi^*(M_0) \le \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} + \sum_{t\in T} E\left[(Y_t - m_t)^+\right]$$

Since $m_t \in [a_t, b_t]$,

$$E\left[(Y_t - m_t)^+\right] = (1 - p_t)(b_t - m_t) = \frac{(b_t - m_t)(m_t - a_t)}{b_t - a_t}$$

which completes the proof.

Q.E.D.

As can be noticed, this upper bound decreases as the bounds of the transition firing times converge, i.e. $b_t \to a_t$, $\forall\, t \in T$. The following theorem shows that it converges to the exact average cycle time which is equal to the average cycle time of the related deterministic timed event graph.

Theorem 10.

Let $s = \sum_{t \in T}(b_t - a_t)$. It holds that :

$$\lim_{s \to 0} \pi(M_0) = \lim_{s \to 0} \pi^*(M_0) = \operatorname*{Max}_{\gamma \in \Gamma} \frac{\sum_{t \in \gamma} m_t}{M_0(\gamma)}$$

The proof of Theorem 10 is similar to that of Theorem 6 and it is omitted.

In a similar way as in section 5, the remainder of this section attempts to improve the previous bound by adequately choosing the two random variable sequences which have been used to derive the upper bound of theorem 5.

Let us consider two sets of random variable sequences $\left\{Y_t^1(k)\right\}_{k=0}^{\infty}$ and $\left\{Y_t^2(k)\right\}_{k=0}^{\infty}$ defined as follows :

$$Y_t^1(k) = z_t$$
$$Y_t^2(k) = \left(Y_t(k) - z_t\right)^+$$

where z_t is a constant belonging to interval $[a_t, b_t]$.

Let $\pi^{1*}(M_0)$ be the average cycle time of STEG1* = $(N, M_0, \{Y_t^1(k)\})$ and $\pi^{2*}(M_0)$ the average cycle time of STEG2* = $(N, M_0, \{Y_t^2(k)\})$.

As in section 4 and in the proof of Theorem 9, it can be shown that

$$\pi^*(M_0) \le \operatorname*{Max}_{\gamma \in \Gamma} \frac{\sum_{t \in \gamma} z_t}{M_0(\gamma)} + \sum_{t \in T} \frac{(m_t - a_t)(b_t - z_t)}{b_t - a_t} \tag{45}$$

Since relation (45) holds for all $z_t \in [a_t, b_t]$, the smallest upper bound can be obtained by adequately choosing the value of z_t as shown in the following theorem.

Theorem 11.

$$\pi(M_0) \le \pi^*(M_0) \le \bar{\pi}(M_0) \tag{46}$$

where

$$\bar{\pi}(M_0) = \underset{a_t \le z_t \le b_t}{\text{Min}} \left\{ \underset{\gamma \in \Gamma}{\text{Max}} \frac{\sum_{t \in \gamma} z_t}{M_0(\gamma)} + \sum_{t \in T} \frac{(m_t - a_t)(b_t - z_t)}{b_t - a_t} \right\} \tag{47}$$

This upper bound can be easily obtained by solving a linear programming problem. Since the upper bound of Theorem 9 can be obtained by taking $z_t = m_t$, the new upper bound is better than the upper bound of Theorem 9. As a result, it also converges to the exact average cycle time as the bounds of the transition firing times converge.

This new bound also shows that the firing time randomness of transitions belonging to elementary circuits with small average cycle times has little effect on the average cycle time of the whole system. The effect of their randomness disappears by taking $z_t = b_t$.

7. CONCLUSION

This paper presents a method for obtaining performance bounds of stochastic timed event graphs. To apply this method, we consider the set of transition firing time sequences as the input of the system. Superposition properties of a stochastic timed event graph, whose input is the superposition of two (or more) sets of random variable sequences, have been established. The most important property is the sub-additivity of the average cycle time which claims that it is smaller than the sum of the average cycle times of the two stochastic timed event graphs with one of the two sets of random variable sequences as input.

This method has first been used to obtain upper bounds of general stochastic timed event graphs. Especially, we have obtained a simple upper bound which converges to the exact average cycle time as the standard deviations decrease.

The improved upper bounds presented in section 5 show that the firing time randomness of transitions belonging to elementary circuits with small average cycle time has little impact on the average cycle time of the net. They also show that in most real-life system, the average cycle time mainly depends on the mean values of the transition firing times.

This method has also been used to stochastic timed event graphs with bounded transition firing times. Upper bounds of the average cycle time which depend on both the bounds of the transition firing times and their means are obtained. These bounds also converge to the exact average cycle time as the bounds of firing times converge.

Further research work includes the extension to other performance measures, the extension to other classes of timed Petri nets and the extension to stochastic models. We believe that similar results can be obtained for generalized semi-Markov models.

BIBLIOGRAPHY

[1] F. BACCELLI, "Ergodic Theory of Stochastic Petri Networks", *Annals of Probability,* Vol. 20, No. 1, pp. 350-374, 1992.

[2] F. BACCELLI and Z. LIU, "Comparison Properties of Stochastic Decision Free Petri Nets", to appear in *IEEE Transactions on Automatic Control,* 1992.

[3] J. CAMPOS, G. CHIOLA and M. SILVA, "Properties and Performance Bounds for Closed Free Choice Synchronized Monoclass Queueing Networks", *IEEE Transactions on Automatic Control,* Vol. 36, No. 12, pp. 1368-1382, December 1991.

[4] P. CHRETIENNE, "Les réseaux de Petri temporisés", Université Paris VI, Paris, France, Thése d'Etat, 1983.

[5] F. COMMONER, A. HOLT, S. EVEN and A. PNUELI: "Marked Directed Graphs", *Journal of Computer and System Science,* Vol. 5, No. 5, pp. 511-523, 1971.

[6] P.J. HAAS and G.S. SHEDLER: "Stochastic Petri Nets: Modeling Power and Limit Theorem", *Probability in Engineering and Informational Sciences,* Vol. 5, pp. 477-498, 1991.

[7] H.P. HILLION and J.M. PROTH: "Performance Evaluation of Job-Shop Systems Using Timed Event-Graphs", *IEEE Transactions on Automatic Control,* Vol. 34, No. 1, pp. 3-9, January 1989.

[8] H.P. HILLION and J.M. PROTH: "Analyse de fabrications non linéaires et répétitives à l'aide des Graphes d'événements Temporisés", *RAIRO,* Vol. 22, No. 2, pp. 137-176, September 1988.

[9] S. LAFTIT, J.M. PROTH and X.L. XIE: "Optimization of Invariant Criteria for Event Graphs", *IEEE Transactions on Automatic Control,* Vol. 37, No. 5, pp. 547-555, May 1992.

[10] T. MURATA, "Petri Nets: Properties, Analysis and Applications," Proceedings of the IEEE, vol. 77, N° 4, pp. 541-580, April 1989.

[11] J.M. PROTH and X.L. XIE: "Performance Evaluation and Optimization of Stochastic Timed Event Graphs", submitted for publication in *IEEE Transactions on Automatic Control,* June 1992.

[12] J.M. PROTH, N. SAUER and X.L. XIE: "Stochastic Timed Event Graphs : Bounds, Cycle Time Reachability and Marking Optimization", submitted for presentation at the 12th IFAC World Congress, 1992.

[13] C.V. RAMAMOORTHY and G.S. HO: "Performance Evaluation of Asynchronous Concurrent Systems using Petri Nets", *IEEE Trans. Software Eng.*, Vol. SE-6, No. 5, pp. 440-449, 1980.

[14] J.A.C. RESING, R.E. de VRIES, G. HOOGHIEMSTRA, M.S. KEANE and G.J. OLSDER: "Asymptotic Behaviour of Random Discrete Event Systems", *Stochastic Processes and their Applications,* Vol. 36, pp. 195-216, 1990.

[15] M. SILVA, "Petri Nets and Flexible Manufacturing," in Advances in Petri Nets 1989, G. Rozenberg (ed.), Lecture Notes of Computer Science, Springer Verlag, pp. 374-417, 1989.

[16] D. STOYAN, "Comparison Methods for Queues and Other Stochastic Models", English translation (D.j. Daley editor), J. Wiley and Sons, New York, 1984.

ISSN 0249 - 6399